Department of Computer & Information Sciences
Pakistan Institute of Engineering and Applied Sciences

Foundations of Logic

Umar Faiz

Foundations of Logic

- Logic a tool for careful reasoning. Mathematical Logic is a tool for working with elaborate compound statements. It includes:
  - A formal language for expressing them.
  - A concise notation for writing them.
  - A methodology for objectively reasoning about their truth or falsity.
  - It is the foundation for expressing formal proofs in all branches of mathematics.

Foundations of Logic: Overview

- Propositional logic:
  - Basic definitions.
  - Equivalence rules & derivations.
- Predicate logic:
  - Predicates.
  - Quantified predicate expressions.
  - Equivalences & derivations.

Propositional Logic

- Propositional Logic is the logic of compound statements built from simpler statements (atomic propositions) using connectives (Boolean).
- Calculus:
  - A method of reasoning by computation of symbols
- Propositional Calculus:
  - Scheme for calculating with logic formulas
- Three different Propositional Calculuses:
  - Semantics-based reasoning – truth tables
  - Syntax-based reasoning – inference rules
  - Equational reasoning – Boolean algebra

Propositional Logic

- The Two Elements of Symbolic Logic:
  - Proposition
  - Logical connectives

Two Elements of Propositional Logic: Propositions

- Propositions:
  - A *proposition* is a statement with a truth value. That is, it is a statement that is true or else a statement that is false. Here are some examples with their truth values.
  - A proposition (denoted p, q, r, …) is simply:
    - a statement (i.e., a declarative sentence)
      - with some definite meaning, (not vague or ambiguous)
    - having a truth value that's either true (T) or false (F)
      - it is never both, neither, or somewhere "in between!"
        - » However, you might not know the actual truth value, and
        - » The truth value might depend on the situation or context.

Two Elements of Propositional Logic: Propositions

- Examples of Propositions:
  - "It is terribly hot outside." (In a given situation.)
  - There are only finitely many prime numbers. (false)
  - "Islamabad is the capital of Pakistan."
  - "4 + 2 = 6"

Two Elements of Propositional Logic: Propositions

- Examples of NOT Propositions:
  - "Who are you?" (interrogative)
  - "Wah Wah" (meaningless exclamation)
  - "Please stop it!" (imperative)
  - "Where is PIEAS?"
  - "Yes, I want to be there" (vague)
  - "2 + 2" (expression with a non-true/false value)
  - Coffee tastes better than tea. (Again, this is a matter of taste, not truth.)
  - Blue is the best color to paint a house.

Two Elements of Propositional Logic: Connectives

- Connectives (Logical Operators):
  - Arithmetic operators (operations) such as addition, subtraction, multiplication, division, and negation act on numbers to give new numbers.
  - Logical operators act on propositions to give new (compound) propositions.
  - The truth value of the compound proposition depends only on the truth value of the component propositions. Such a list is a called a truth table.

Two Elements of Propositional Logic: Connectives

- Connectives (Logical Operators):

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

Two Elements of Propositional Logic: Connectives

1. The Negation Operator:
   - The unary negation operator "¬" (NOT) transforms a prop. into its logical negation.
     - e.g. If p = "I have brown hair."
     - then ¬p = "I do not have brown hair."
   - The truth table for NOT:

     T :≡ True; F :≡ False

     ":≡" means "is defined as"

p	$\neg p$
T	F
F	T
Operand column	Result column

Two Elements of Propositional Logic: Connectives

1. Negation Truth Table:

p	$\neg p$
F	T
T	F

T :≡ True; F :≡ False

":≡" means "is defined as"

Two Elements of Propositional Logic: Connectives

2. The Conjunction Operator:

– The binary conjunction operator "∧" (AND) combines two propositions to form their logical conjunction.
  - e.g. If p="I will have salad for lunch." and q="I will have steak for dinner.", then p∧q="I will have salad for lunch and AND I will have steak for dinner."

Two Elements of Propositional Logic: Connectives

2. Conjunction Truth Table:

p	*q*	*p*∧*q*
T	T	T
T	F	F
F	T	F
F	F	F

– True iff <u>both</u> operands are True
  - Define P = x > 0, Q = x < 10
  - P ∧ Q is True iff x is between 0 and 10

Two Elements of Propositional Logic: Connectives

3. The Disjunction Operator:

– The binary disjunction operator "∨" (OR) combines two propositions to form their logical disjunction.
  - p="My car has a bad engine."
  - q="My car has a bad carburetor."
  - p∨q="Either my car has a bad engine, or my car has a bad carburetor."

Two Elements of Propositional Logic: Connectives

3. Disjunction Truth Table:

– p∨q means that p is true, or q is true, or both are true. This operation is also called inclusive or, because it includes the possibility that both p and q are true.

p	*q*	*p*∨*q*
T	T	T
T	F	T
F	T	T
F	F	F

Two Elements of Propositional Logic: Connectives

Exercise:

- Let p="It rained last night",
  q="The sprinklers came on last night,"
  r="The lawn was wet this morning."
- Translate each of the following into English:

¬p = "It didn't rain last night."

r ∧ ¬p = "The lawn was wet this morning, and it didn't rain last night."

¬ r ∨ p ∨ q = "Either the lawn wasn't wet this morning, or it rained last night, or the sprinklers came on last night."

Two Elements of Propositional Logic: Connectives

4. The Exclusive Or Operator:

– The binary exclusive-or operator "⊕" (XOR) combines two propositions to form their logical "exclusive or" (exjunction?).
  - p = "I will earn an A in this course,"
  - q = "I will drop this course,"
  - p ⊕ q = "I will either earn an A in this course, or I will drop it (but not both!)"

Two Elements of Propositional Logic: Connectives

4. Exclusive-Or Truth Table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- This operation is called exclusive or, because it excludes the possibility that both p and q are true.

19

Two Elements of Propositional Logic: Connectives

Natural Language is Ambiguous:

- Note that English "or" can be ambiguous regarding the "both" case!
- "`Shamim is a singer or Shamim is a writer." - $\vee$
- "Shamim is a man or Shamim is a woman." - $\oplus$

p	q	p "or" q
F	F	F
F	T	T
T	F	T
T	T	?

- Need context to disambiguate the meaning!

20

Two Elements of Propositional Logic: Connectives

5. The Implication Operator:

- The implication p → q states that p implies q.
  - i.e., If p is true, then q is true; but if p is not true, then q could be either true or false.
    - e.g., let p = "You study hard."
      q = "You will get a good grade."
  - p → q = "If you study hard, then you will get a good grade." (else, it could go either way)

21

Two Elements of Propositional Logic: Connectives

Implication Truth Table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- p → q is false only when p is true but q is not true.

22

Conditional: why F→F is True

- Mathematically, p should imply q whenever it is possible to derive q by from p by using valid arguments. For example consider the following:
  - If 0 = 1 then 3 = 9.
- Q: Is this true mathematically?

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)

3. 3 = 6 (multiplied both sides by 3)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)
3. 3 = 6 (multiplied both sides by 3)

4. 0 = 3 (multiplied no. 1 by 3)

Conditional: why F→F is True

A: YES mathematically and YES by the truth table.

Here's a mathematical proof:

1. 0 = 1 (assumption)
2. 1 = 2 (added 1 to both sides)
3. 3 = 6 (multiplied both sides by 3)
4. 0 = 3 (multiplied no. 1 by 3)

5. 3 = 9 (added no. 3 and no. 4) *QED*

Two Elements of Propositional Logic: Connectives

Examples of Implications:

- "If this lecture ever ends, then the sun will rise tomorrow." True or False?
- "If Tuesday is a day of the week, then I am a penguin." True or False?
- "If 1+1=6, then Bush is president." True or False?
- "If the moon is made of green cheese, then I am richer than Bill Gates." True or False?

Two Elements of Propositional Logic: Connectives

English Phrases Meaning p → q

- "p implies q"
- "if p, then q"
- "if p, q"
- "when p, q"
- "whenever p, q"
- "q if p"
- "q when p"
- "q whenever p"
- "*p* only if *q*"
- "*p* is sufficient for *q*"
- "*q* is necessary for *p*"
- "*q* follows from *p*"
- "*q* is implied by *p*"

Two Elements of Propositional Logic: Connectives

Converse, Inverse, Contrapositive:

- For an implication $p \to q$:
  - Its converse is: $q \to p$.
  - Its inverse is: $\neg p \to \neg q$.
  - Its contrapositive: $\neg q \to \neg p$.
  - One of these three has the same meaning (same truth table) as $p \to q$. Can you figure out which?

Two Elements of Propositional Logic: Connectives

- Proving the equivalence of $p \to q$ and its contrapositive using truth tables:

p	q	$\neg q$	$\neg p$	$p \to q$	$\neg q \to \neg p$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

Two Elements of Propositional Logic: Connectives

6. The Biconditional Operator:

The biconditional $p \leftrightarrow q$ states that p is true if and only if (IFF) q is true.

- p = "Bush wins the 2004 election."
- q = "Bush will be president for all of 2005."
- $p \leftrightarrow q$ = "If, and only if, Bush wins the 2004 election, Bush will be president for all of 2005."

Two Elements of Propositional Logic: Connectives

- Biconditional Truth Table:
- $p \leftrightarrow q$ means that p and q have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- $p \leftrightarrow q$ does not imply that p and q are true, or that either of them causes the other, or that they have a common cause.

Two Elements of Propositional Logic: Connectives

- Boolean Connectives Summary:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

Two Elements of Propositional Logic: Connectives

- Some Alternative Notations:

Name:	not	and	or	xor	implies	iff
Propositional logic:	$\neg$	$\wedge$	$\vee$	$\oplus$	$\to$	$\leftrightarrow$
Boolean algebra:	$\bar{p}$	pq	+	$\oplus$		
C/C++/Java (wordwise):	!	&&	||	!=		==
C/C++/Java (bitwise):	~	&	|	^		
Logic gates:						

Bits and Bit Operations

- A bit is a binary (base 2) digit: 0 or 1.
- Bits may be used to represent truth values.
- By convention:
  0 represents "false"; 1 represents "true".
- Boolean algebra is like ordinary algebra except that variables stand for bits, + means "or", and multiplication means "and"

37

Bitwise Operations

- Boolean operations can be extended to operate on bit strings as well as single bits.
  - e.g.:

```
01 1011 0110
11 0001 1101
11 1011 1111  Bit-wise OR
01 0001 0100  Bit-wise AND
10 1010 1011  Bit-wise XOR
```

38

Recap

- You have learned about:
- Propositions: What they are?
- Propositional logic operators
  - Symbolic notations.
  - English equivalents.
  - Logical meaning.
- Truth tables.
- Atomic vs. Compound propositions.
- Alternative notations.
- Bits and bit-strings.

- Next lecture:
  - Propositional equivalences.
  - How to prove them.

39